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Positive Linear Operators with Equidistant Nodes

I. GAVREA
Department of Mathematics, Technical University of Cluj-Napoca
RO-3400 Cluj-Napoca, Romania

H. H. GONSKA AND D. P. KACSÓ*
Department of Mathematics, University of Duisburg
D-47048 Duisburg, Germany
<gonska><kacso>@informatik.uni-duisburg.de

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Abstract—In the present paper, the approximation power of positive linear operators with equidistant nodes is investigated. New pointwise estimates are given in terms of first and second order moduli of continuity, showing that for positive operators having uniform Jackson orders of approximation one may expect interpolation at the endpoints. In particular, the first solution to Butzer's problem with equidistant nodes is given. A negative result yields an explanation why Bernstein operators and some of their modifications are optimal in a certain sense.

Keywords—Positive linear operators, Degree of approximation, First and second order moduli of continuity, Endpoint interpolation, Butzer's problem.

1. INTRODUCTION

In 1992, Gonska and Zhou [1] formulated the following problem.

PROBLEM #1. "Do there exist *positive* linear operators $L_{rn+s} : C[0,1] \to \Pi_{rn+s}$, $r \in \mathbb{N}$, $s \in \mathbb{Z}$ fixed, such that

$$|L_{rn+s}(f;x) - f(x)| \le c \cdot \omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \tag{1}$$

for $f \in C[0,1]$, $x \in [0,1]$ and the constant c independent of f, x, and n?"

The following weaker requests were also formulated in [1]: "Do there exist operators of the type described above such that

$$|L_{rn+s}(f;x) - f(x)| \le c \cdot \omega_1\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \tag{2}$$

or such that

$$|L_{rn+s}(f;x) - f(x)| \le c \cdot \omega_2\left(f; \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}\right), \tag{3}$$

with all quantities specified as in the above?"

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*On leave from: Department of Mathematics, Technical University of Cluj-Napoca, RO-3400 Cluj-Napoca, Romania.

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In formulating these questions, the aforementioned authors were guided by

- a result of DeVore [2] stating, roughly speaking, that in a Jackson-type inequality for algebraic polynomial approximation it is possible to have an upper bound as given in (1), and by
- a problem formulated by Butzer [3] in 1980, in which he posed the problem to construct discretely defined *positive* linear operators giving an approximation order of $\mathcal{O}(n^{-\alpha})$ for $f \in \mathrm{Lip}_2\alpha$, $0 < \alpha \leq 2$.

Linear polynomial operators satisfying the so-called DeVore-Gopengauz inequalities, that is, *arbitrary* linear polynomial operators satisfying (1), were investigated in a sequence of papers by Cao and Gonska (see, e.g., [4] and the references cited there). However, due to the Boolean sum approach used in their papers, the operators from there are nonpositive. Furthermore, the first solution to Butzer's problem was given in 1989 by the same authors [5], but the discretely defined positive linear operators constructed there do not interpolate at the endpoints, and their positivity would be destroyed using the Boolean sum technique in order to impose interpolation at the endpoints.

In 1993, Gavrea [6] was the first to construct a sequence of positive linear operators satisfying (2), that is, a positive solution to Problem #1, but only in terms of ω_1.

In his talk at the 1995 International Dortmund Meeting on Approximation Theory, the second author summarized the (then) state of the art using a graph similar to the one given below. Since it was unknown in 1995 if there is a sequence of positive linear operators satisfying (1), some open problems for *positive* operators were summarized as follows. Let

$$\varphi_1(x,n) = \frac{\sqrt{x(1-x)}}{n}, \qquad \varphi_2(x,n) = \frac{\sqrt{x(1-x)}}{n} + \frac{1}{n^2}, \qquad \text{and} \qquad \varphi_3(x,n) = \frac{1}{n}$$

be the "classical" second parameters in ω_2-estimates. Each of the vertices (= boxes) of the digraph below represented an open question. An arrow from box $\boxed{\text{A}}$ to box $\boxed{\text{B}}$ meant that a positive answer to the question represented by box $\boxed{\text{A}}$ would automatically lead to a positive answer to the question represented by box $\boxed{\text{B}}$, but not vice versa. Furthermore, it should be read as follows.

A box containing the string "ω_2: general, φ_i?" represents the question if there are *any* positive linear operators satisfying (1) with φ_1 replaced by φ_i. Likewise, the box having "ω_2: discrete, φ_i?" in it stands for the problem if there are *discrete* positive linear operators doing the same, and "ω_2: equidistant, φ_i?" also asks for discretely defined operators, but with *equidistant* nodes.

Only recently, Gavrea [7] also solved "ω_2: general, φ_1?" by constructing positive linear operators satisfying the inequality in question. We will return to this construction in Section 3. Moreover, the three authors of this paper succeeded in giving a positive solution to "ω_2: discrete, φ_1?" (see [8]).

The question represented by "ω_2: equidistant, φ_3?" is answered in the affirmative in Section 3 of the present paper.

There are two boxes missing from the Figure 1, namely the ones for "ω_2: equidistant, φ_i?," $i = 1, 2$. However, here the answer is negative, which follows from a result of Vértesi [9] treating even the analogous questions for the ω_1-case. A similar graph can be drawn with ω_2 replaced by ω_1. Two earlier results to be mentioned here are the one of Szabados [10] who showed in 1976 that the problem representable by "ω_1: equidistant, φ_3?" has a positive solution (thus solving a problem of DeVore), and the one by Gavrea [6] where "ω_1: general, φ_1?" was solved for the first time. Of course, our positive answer to "ω_2: discrete, φ_1?" implies a solution for "ω_1: discrete, φ_1?".

The present paper supplements the information available in the following sense. For the ω_2-case. a first solution to "ω_2: equidistant, φ_3?" is given. Thus, in combination with our other papers mentioned before, all the problems represented by Figure 1 will be solved. In fact, we will prove

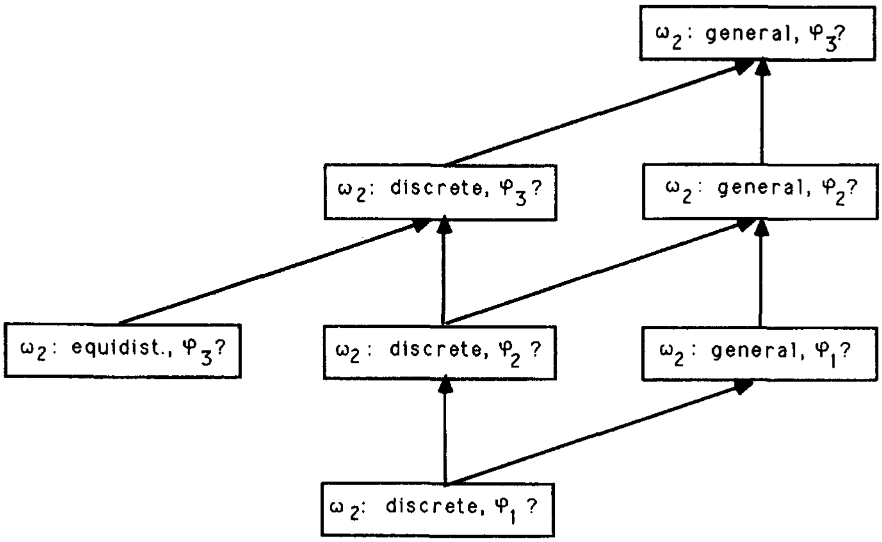

Figure 1. Open problems concerning ω_2 estimates for positive linear operators: the state of the art early in 1995.

an even stronger result in the ω_2-case by giving a pointwise estimate implying interpolation at the endpoints of $[0,1]$. This result is also the first solution to Butzer's problem which is based on operators with equidistant nodes and which is derived from an inequality having only ω_2 in its upper bound. In regard to the ω_1-case, in Section 2 a strengthening of the result of Szabados will be given by showing that there exists a positive solution to "ω_1: equidistant, φ_3?" exhibiting interpolation at the endpoints. Although this is also implied by our positive assertion of Section 3, the construction in Section 2 is different and should thus be of independent interest. Section 3 also contains a negative result giving a more careful analysis of how close one can get to a solution of "ω_2: equidistant, φ_2?".

2. THE ω_1-CASE

We start off with the description of a construction by Lupaş published in 1995 [11]. Let $D_n^{\langle\alpha\rangle}$ be the Durrmeyer operator defined by

$$\left(D_n^{\langle\alpha\rangle} f\right)(x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 t^\alpha (1-t)^\alpha p_{n,k}(t) f(t)\,dt}{\int_0^1 t^\alpha (1-t)^\alpha p_{n,k}(t)\,dt} \tag{4}$$

for $f \in C[0,1]$, with $\alpha > -1$.

Let Q_n be a polynomial of degree n given by

$$Q_n(x) = \sum_{k=0}^{n} a_{k,n} x^k, \qquad a_{n,n} \neq 0,$$

satisfying the following conditions:

$$Q_n(x) \geq 0, \qquad \forall x \in [0,1] \text{ and } \int_0^1 Q_n(x)\,dx = 1. \tag{5}$$

Lupaş considered the sequence of linear polynomial operators $L_n^{\langle\alpha\rangle} : C[0,1] \to \Pi_n$, where

$$\left(L_n^{\langle\alpha\rangle} f\right)(x) = \sum_{k=0}^{n} \frac{(\alpha+1)_k}{(2\alpha+2)_k} a_{k,n} \left(D_k^{\langle\alpha\rangle} f\right)(x), \tag{6}$$

with $(\alpha)_k = \alpha\,(\alpha+1)\dots(\alpha+k-1)$ for $k \geq 1$, and $(\alpha)_0 = 1$. He proved that if Q_n verifies the conditions (5), then the operators $(L_n^{\langle\alpha\rangle})_{n\in\mathbb{N}}$ are positive. Let $d := d(n) := n - 2[n/2]$ and $s := s(n) := 1 + [n/2]$. Taking

$$Q_n(x) = \lambda_n^* x^d \left(\frac{J_s^{(\alpha,\alpha+d)}(x)}{x - x_{1,n}^{(\alpha)}}\right)^2, \tag{7}$$

where $J_s^{(\alpha,\beta)}$ is the Jacobi polynomial of degree s relative to the interval [0,1] and $x_{i,s}^{(\alpha)}$, $1 \leq i \leq s$, satisfying $0 < x_{s,s}^{(\alpha)} < x_{s-1,s}^{(\alpha)} < \cdots < x_{1,s}^{(\alpha)} < 1$, are the roots of $J_s^{(\alpha,\alpha+d)}$, Lupaş showed that the operators $(L_n^{\langle\alpha\rangle})_{n\in\mathbb{N}}$ given in (6) satisfy the inequality

$$\left|f(x) - \left(L_n^{\langle\alpha\rangle} f\right)(x)\right| \leq c\,\omega_1\left(f;\ \Delta_n(x)\right),$$

with $\Delta_n(x) = 1/n^2 + (\sqrt{x(1-x)})/n$, $|\alpha| \leq 1/2$, and the constant c independent on f, x, and n.

Here we mention that in this paper the constant c denotes a positive constant which can be different at each occurrence.

For $f \in C[0,1]$, let us now consider the Balász-Szabados (Shepard-type) operators $(R_n)_{n\in\mathbb{N}}$ defined by

$$(R_n\, f)(x) = \frac{\sum_{k=0}^n f(k/n)(x-(k/n))^{-4}}{\sum_{k=0}^n (x-(k/n))^{-4}},$$

and Lupaş' operators $S_n^{\langle\alpha\rangle} = L_n^{\langle\alpha\rangle} R_n : C[0,1] \to \Pi_n$, where $L_n^{\langle\alpha\rangle}$ is given as in (6) with Q_n from (7).

For the latter operators $S_n^{\langle\alpha\rangle}$, in [11] Lupaş gave the following estimate:

$$\left\|f - S_n^{\langle\alpha\rangle} f\right\| \leq c\cdot\omega_1\left(f; \frac{1}{n}\right), \qquad \text{for } f \in C[0,1] \text{ and } |\alpha| \leq \frac{1}{2}, \tag{8}$$

and with the constant c neither depending on f nor on n. The operators $(S_n^{\langle\alpha\rangle})_{n\in\mathbb{N}}$ are linear and positive.

Using the operators $S_n^{\langle\alpha\rangle}$ we can now prove the following theorem.

THEOREM 1. *There exists a sequence of linear positive operators $H_{5n-2} : C[0,1] \to \Pi_{5n-2}$ of the form*

$$(H_{5n-2} f)(x) = \sum_{k=0}^n q_{5n-2,k}(x) f\left(\frac{k}{n}\right), \tag{9}$$

such that the estimate

$$|H_{5n-2}(f;x) - f(x)| \leq c\cdot\omega_1\left(f; \frac{1-P_{2n-1}^2(x)}{n} + \frac{\sqrt{x(1-x)}}{n}\right) \tag{10}$$

holds for all $x \in [0,1]$, where P_{2n-1} is the Legendre polynomial of degree $2n-1$ relative to the interval $[0,1]$ and normalized such that $P_{2n-1}(1) = 1$.

The inequality of Theorem 1 can also be read as in the following corollary.

COROLLARY 2. *There exists a sequence of linear positive operators $H_{5n-2} : C[0,1] \to \Pi_{5n-2}$ of the form (9) such that one has*

$$|H_{5n-2}(f;x) - f(x)| \leq c\cdot\omega_1\left(f; \frac{\alpha_n(x)}{n} + \frac{\sqrt{x(1-x)}}{n}\right),$$

for all $x \in [0,1]$, *where* $\alpha_n \in C[0,1]$ *is a uniformly bounded function satisfying* $\alpha_n(0) = \alpha_n(1) = 0$.

PROOF OF THEOREM 1. We consider the operator $H_{5n-2} : C[0,1] \to \Pi_{5n-2}$ defined by

$$H_{5n-2}(f;x) = \left(1 - P_{2n-1}^2(x)\right) S_n^{\langle\alpha\rangle}(f;x) + P_{2n-1}^2(x) \cdot [(1-x)f(0) + xf(1)], \tag{11}$$

where $S_n^{\langle\alpha\rangle} = L_n^{\langle\alpha\rangle} R_n$ are Lupaş' operators.

Because $P_{2n-1}^2 \le 1$, it is obvious that H_{5n-2} is a positive operator reproducing constants.

It is well known that for every function $f \in C[0,1]$, one has

$$|H_{5n-2}(f;x) - f(x)| \le 2\omega_1\left(f; H_{5n-2}(|t-x|;x)\right).$$

But $H_{5n-2}(|t-x|;x) = (1 - P_{2n-1}^2(x))\, S_n^{\langle\alpha\rangle}(|t-x|;x) + P_{2n-1}^2(x) \cdot 2x(1-x)$.

Using the Bernstein inequality (see [12])

$$|P_n(x)| \le \sqrt{\frac{2}{n\pi}}\,[x(1-x)]^{-1/4}, \qquad \text{for all } x \in (0,1)$$

and the inequality (8), we get the following upper bound for $H_{5n-2}(|t-x|;\, x)$:

$$\begin{aligned} H_{5n-2}(|t-x|x) &\le \left(1 - P_{2n-1}^2(x)\right)\frac{c}{n} + \frac{2}{\pi}\frac{\sqrt{x(1-x)}}{n} \\ &\le c\left[\frac{1 - P_{2n-1}^2(x)}{n} + \frac{\sqrt{x(1-x)}}{n}\right]. \end{aligned} \tag{12}$$

Because the operators H_{5n-2} are of the form (9), it follows that the assertion of Theorem 1 is proved. ∎

The proof of Corollary 2 now is immediate after putting $\alpha_n(x) = 1 - P_{2n-1}^2(x)$ and observing that $\|\alpha_n\| \le 1$, for all $n \in \mathbb{N} \setminus \{0\}$.

One further consequence of Theorem 1 is given as the following corollary.

COROLLARY 3. *There exists a sequence of positive linear operators* $H_{5n-2} : C[0,1] \to \Pi_{5n-2}$ *defined as in (9), such that for all* $x \in [0,1]$ *one has*

$$|H_{5n-2}(f;x) - f(x)| \le c \cdot \omega_1\left(f; \min\left(\frac{[x(1-x)]^{\beta/2}}{n^{1-2\beta}}, \frac{[x(1-x)]^{\beta/4}}{n^{1-\beta}}\right) + \frac{\sqrt{x(1-x)}}{n}\right) \tag{13}$$

with $\beta \in [0,1]$.

PROOF. In order to prove Corollary 3, we need the following lemma given in [13].

LEMMA 4. *Let* P_{2n-1} *be the Legendre polynomial of degree* $2n-1$ *relative to the interval* $[0,1]$ *and normalized such that* $P_{2n-1}(1) = 1$. *Then the inequality*

$$1 - P_{2n-1}^2(x) \le (2n-1)\frac{\pi}{\sqrt{2}}\sqrt[4]{x(1-x)}$$

is satisfied. ∎

For all $\beta \in [0,1]$, one has

$$1 - P_{2n-1}^2(x) \le \left(1 - P_{2n-1}^2\right)^{\beta}.$$

Using Lemma 4, we get

$$1 - P_{2n-1}^2(x) \le 2^{\beta/2}\pi^{\beta} n^{\beta}[x(1-x)]^{\beta/4}.$$

Now the upper bound of $H_{5n-2}(|t-x|;x)$ from (12) becomes

$$H_{5n-2}(|t-x|;x) \le c\left\{\frac{[x(1-x)]^{\beta/4}}{n^{1-\beta}} + \frac{\sqrt{x(1-x)}}{n}\right\}. \tag{14}$$

On the other hand, we can write

$$1 - P_{2n-1}^2(x) = \int_0^x P'_{2n-1}(t)\,dt \cdot \int_x^1 P'_{2n-1}(t)\,dt.$$

Now using the following inequality due to Bernstein,

$$\frac{|P'(x)|}{\max_{x\in[a,b]}|P(x)|} \le \frac{n}{\sqrt{(b-x)(x-a)}}, \qquad P \in \Pi_n, \quad x \in (a,b),$$

we obtain

$$1 - P_{2n-1}^2(x) \le (2n-1)^2 \int_0^x \frac{dt}{\sqrt{t(1-t)}} \cdot \int_x^1 \frac{dt}{\sqrt{t(1-t)}} = (2n-1)^2\left[\frac{\pi^2}{4} - \arcsin^2(2x-1)\right].$$

But $\pi^2/4 - \arcsin(2x-1) \le \pi^2/2\sqrt{x(1-x)}$ (see [13]), so it follows that

$$1 - P_{2n-1}^2(x) \le \pi^{2\beta} \cdot 2^\beta \cdot n^{2\beta} \cdot [x(1-x)]^{\beta/2}.$$

Thus (12) now implies

$$H_{5n-2}(|t-x|;x) \le c\left\{\frac{[x(1-x)]^{\beta/2}}{n^{1-2\beta}} + \frac{\sqrt{x(1-x)}}{n}\right\}. \tag{15}$$

The inequalities (14) and (15) lead to

$$H_{5n-2}(|t-x|;x) \le c \cdot \left\{\min\left(\frac{[x(1-x)]^{\beta/2}}{n^{1-2\beta}}, \frac{[x(1-x)]^{\beta/4}}{n^{1-\beta}}\right) + \frac{\sqrt{x(1-x)}}{n}\right\}$$

which completes the proof. ∎

The results given in Theorem 1 and its corollaries show that it is possible to have interpolation at the endpoints for operators solving "ω_1: equidistant, φ_3?". This cannot be read off Szabados' solution mentioned earlier. However, the problems "ω_1: equidistant, φ_2?" and "ω_1: equidistant, φ_1?" have negative answers, and thus the theorem and the corollaries also show, in a sense, how to bridge the gap between the possible and the impossible.

3. THE ω_2-CASE

In this section, a result similar to that of Corollary 2 is given, but for the ω_2-case. However, it is of interest also, since it is based on a construction different from that in Theorem 1. We note that the first solution to Butzer's problem is given below, which is based on operators with equidistant nodes and which is derived from an inequality having only ω_2 in its upper bound. In particular, this is also the first positive solution to "ω_2: equidistant, φ_3?". Our technique employed here is related to Gavrea's solution of the problem "ω_2: general, φ_1?". He constructed a sequence of positive linear operators $H_{2n+1} : C[0,1] \to \Pi_{2n+1}$, defined in the following way.

Let $L_n : C[0,1] \to \Pi_n$ be of the form

$$(L_n f)(x) = f(0)(1-x)^n + x^n f(1) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t) f(t)\,dt,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n = 0, 1, \dots$.

We consider the Jacobi polynomial $J_n^{(1,0)}$ relative to the interval [0,1], normalized by the condition $J_n^{(1,0)}(1) = 1$, and the polynomial

$$P_{2n-1}(x) = \lambda_n \int_0^x \left(\frac{J_n^{(1,0)}(t)}{t - x_n} \right)^2 dt = \sum_{k=1}^{2n-1} a_k x^k,$$

where

$$\lambda_n = \frac{1}{\int_0^1 (1-x) \left(\frac{J_n^{(1,0)}(x)}{x - x_n} \right)^2 dx},$$

and x_n is the largest root of the polynomial $J_n^{(1,0)}$.

Then the operators H_{2n+1} are given by

$$H_{2n+1} f = \sum_{k=0}^{2n-1} \frac{a_k}{k+1} L_{k+2} f. \tag{16}$$

For these operators, Gavrea proved that

$$|(H_{2n+1} f)(x) - f(x)| \le c\omega_2 \left(f; \frac{\sqrt{x(1-x)}}{n} \right). \tag{17}$$

Using this construction we can now show the following theorem.

THEOREM 5. *There exists a sequence of linear positive operators $T_{2n+1} : C\,[0,1] \to \Pi_{2n+1}$ of the form*

$$(T_{2n+1} f)(x) = \sum_{k=0}^{n} q_{2n+1,k}(x) f\left(\frac{k}{n}\right)$$

satisfying the inequality

$$|T_{2n+1}(f;x) - f(x)| \le c \cdot \omega_2 \left(f; \frac{\alpha_n(x)}{n} + \frac{\sqrt{x(1-x)}}{n} \right), \tag{18}$$

where $\alpha_n \in C\,[0,1]$ is a uniformly bounded function such that $\alpha_n(0) = \alpha_n(1) = 0$.

PROOF. We consider the operator

$$(S_n f)(x) = \frac{1}{n} \sum_{k=0}^{n} \left[\frac{k-1}{n}, \frac{k}{n}, \frac{k+1}{n};\ |t - x| \right]_t \cdot f\left(\frac{k}{n}\right),$$

where, for mutually distinct a, b, c, we denote by $[a, b, c; f(t,x)]_t$ the fact that the divided difference is applied on the variable t.

This operator has the following properties (see [14]):

(i) $S_n : C\,[0,1] \to C\,[0,1]$ is positive and linear,
(ii) $(S_n f)(k/n) = f(k/n)$, $k = 0, \dots, n$,
(iii) $S_n e_i = e_i$, $i = 0, 1$,
(iv) $S_n(\Omega_{x,1})(x) = (z_n(x)(1 - z_n(x)))/n$,
$S_n(\Omega_{x,2})(x) = (z_n(x)(1 - z_n(x))/n^2$,

where $\Omega_{x,i}(t) = |t - x|^i$, $i \in \mathbb{N}$ and $z_n(x) = nx - [nx]$.

Note that $S_n f$ is the piecewise linear continuous function that interpolates f at the points k/n, $k = 0, \dots, n$.

We define now the operators $T_{2n+1} = H_{2n+1}\, S_n$, and we will prove in the sequel that these operators satisfy the inequality (18) from the theorem. We have

$$T_{2n+1}\left(\Omega_{x,2}(y)\right)(x) = H_{2n+1}\left(S_n(y-u)^2;\, u\right)(x) + H_{2n+1}\left((u-x)^2\right)(x).$$

Using property (iv), we obtain

$$T_{2n+1}\left(\Omega_{x,2}(y)\right)(x) = H_{2n+1}\left(\frac{z_n(u)\left(1-z_n(u)\right)}{n^2}\right)(x) + H_{2n+1}\left((u-x)^2\right)(x). \tag{19}$$

We will need here the following result established by Gonska and Kovacheva in [15]. If L is a positive linear operator defined on $C(I)$ with $Le_0 = e_0$, $Le_1 = e_1$, then for $f \in C(I)$, $x \in I$ and each h, $h \in (0,(1/2)]$, we have

$$|(Lf)(x) - f(x)| \leq \left[\frac{3}{2} + \frac{3}{4}h^{-2}\, L\left((e_1 - x)^2; x\right)\right] \omega_2(f;h).$$

We can thus write

$$|(T_{2n+1}f)(x) - f(x)| \leq \frac{9}{4}\omega_2\left(f; \sqrt{T_{2n+1}\left(\Omega_{x,2}\right)}\right). \tag{20}$$

Using the relation (19) and the fact that

$$\left(H_{2n+1}\Omega_{x,2}\right)(x) \leq c \cdot \frac{x(1-x)}{n^2},$$

we get the upper bound

$$\sqrt{\left(T_{2n+1}\Omega_{x,2}\right)(x)} \leq c\left(\frac{\alpha_n(x)}{n} + \frac{\sqrt{x(1-x)}}{n}\right),$$

where $\alpha_n(x) = \sqrt{H_{2n+1}\left(z_n\left(1-z_n\right)\right)(x)}$.

Thus, the latter inequality together with (20) lead us to the estimate

$$|(T_{2n+1}f)(x) - f(x)| \leq c \cdot \omega_2\left(f; \frac{\alpha_n(x)}{n} + \frac{\sqrt{x(1-x)}}{n}\right).$$

Because $(H_{2n+1}(z_n(1-z_n)))(0) = (H_{2n+1}(z_n(1-z_n)))(1) = 0$ and $0 \leq z_n(1-z_n) \leq 1/4$, we obtain the statement of the theorem. ∎

The result of Theorem 5 raises the question what else one can expect from positive linear operators with equidistant nodes. Our next theorem shows that the problem represented by "ω_2: equidistant, φ_2?" has a negative answer. Hence, this is also the case for "ω_2: equidistant, φ_1?". Although this is also a consequence of Vértesi's result mentioned earlier, the following assertion contains an even stronger result.

THEOREM 6. *There do not exist positive linear operators $L_n : C[0,1] \to C[0,1]$ given by*

$$\left(L_n f\right)(x) = \sum_{k=0}^{n} q_{n,k}(x) f\left(\frac{k}{n}\right)$$

and satisfying the estimate

$$|L_n(f;x) - f(x)| \leq c \cdot \omega_2\left(f; \frac{\sqrt{x(1-x)}}{n^a} + \frac{\alpha_n}{n^b}\right) \tag{21}$$

with $a \geq 1/2 + \varepsilon$ and $b \geq 3/2 + \varepsilon_1$, $\varepsilon > 0$, $\varepsilon_1 \geq 0$ and $|\alpha_n| \leq M$.

PROOF. We write $x_{k,n} = k/n$. Assume that there exist operators $L_n : C[0,1] \to C[0,1]$ satisfying the hypothesis of the theorem. Then the following conditions hold:

$$q_{n,i}(x) \geq 0, \qquad \text{for } i = 0, 1, \dots, n,$$
$$\sum_{k=1}^{n} q_{n,k}(x) \cdot x_{k,n} = x, \text{ and}$$
$$(L_n e_2)(x) \leq x^2 + 2c \left(\frac{\sqrt{x(1-x)}}{n^a} + \frac{\alpha_n}{n^b} \right)^2. \tag{22}$$

We have

$$(L_n e_2)(x) = \sum_{k=1}^{n} q_{n,k}(x) \cdot x_{k,n}^2 \geq x_{1,n} \cdot \sum_{k=1}^{n} q_{n,k} \cdot x_{k,n} = x_{1,n} \cdot x.$$

Thus, the inequality (22) becomes

$$x \cdot x_{1,n} \leq x^2 + 2c \left(\frac{\sqrt{x(1-x)}}{n^a} + \frac{\alpha_n}{n^b} \right)^2, \qquad \text{for all } x \in [0,1].$$

Taking $x = 1/n^2$ in the latter inequality, we obtain

$$x_{1,n} \cdot \frac{1}{n^2} \leq \frac{1}{n^4} + c \left(\frac{\sqrt{1 - 1/n^2}}{n^{a+1}} + \frac{\alpha_n}{n^b} \right)^2.$$

For $a = 1/2 + \varepsilon$ and $b = 3/2 + \varepsilon_1$, this becomes

$$x_{1,n} \leq \frac{1}{n^2} + c \cdot n^2 \left(\frac{\sqrt{1 - 1/n^2}}{n^{3/2+\varepsilon}} + \frac{\alpha_n}{n^{3/2+\varepsilon_1}} \right)^2.$$

W.l.o.g., we may suppose $\varepsilon > \varepsilon_1$. We will treat separately the cases $\varepsilon_1 > 0$ and $\varepsilon_1 = 0$. We handle first the case $\varepsilon_1 > 0$. This leads us to the following:

$$x_{1,n} \leq \frac{1}{n^2} + \frac{c}{n^{1+2\varepsilon_1}}.$$

The latter inequality implies that

$$\lim_{n \to \infty} n \cdot x_{1,n} = 0.$$

But $\lim_{n \to \infty} n \cdot x_{1,n} = \lim_{n \to \infty} n \cdot 1/n = 1$, so we have a contradiction.

For the case $\varepsilon_1 = 0$, it follows from inequality (22) that

$$x \cdot x_{1,n} \leq x^2 + 2c \left(\frac{\sqrt{x(1-x)}}{n^{1/2+\varepsilon}} + \frac{\alpha_n}{n^{3/2}} \right)^2.$$

Taking $x = 1/n^{3/2}$ in the latter inequality, we get

$$x_{1,n} \leq \frac{1}{n^{3/2}} + \frac{2c}{n^{1+2\varepsilon}}.$$

This implies that $\lim_{n \to \infty} n \cdot x_{1,n} = 0$, leading us again to a contradiction. ∎

REMARK 7. Looking back at the estimates of Theorem 6, all cases with $\varepsilon > 0$ are clearly covered by that result. For the extremal case $\varepsilon = 0$ (not covered by Theorem 6), of course, there exist positive linear operators with equidistant point evaluations: the classical Bernstein operator, their Stancu modification $L_{m,0}^{\langle \alpha_m,0,0\rangle}$ with $\alpha_m = \mathcal{O}(1/m)$ (see [16], for example), or the modified Bernstein operators $P_{n,s}$ of Stancu investigated in [17]. For all operators L_n mentioned before, one has inequalities of the type

$$|(L_n f - f)(x)| \leq c \cdot \omega_2 \left(f; \frac{\sqrt{x(1-x)}}{n^{1/2}} \right) \tag{23}$$

with c independent of f, x, and n; but still depending on the sequence (L_n).

Clearly enough, these operators also satisfy (23) with a positive term of the form α_n/n^b added. But it does not make sense to discuss this any further, since Theorem 6 is showing that Bernstein operators and some of their Stancu modifications are indeed the "perfect equidistant tools" to have inequalities of the type (23).

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